On distributive laws in derived bracket construction

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Abstract

We will introduce new relations which naturally arises between Lie-operad and Leibniz-operad. We prove that the relations are "distributive laws" of the Lie operad over the Leibniz operad in the sense of M. Markl (1996). By using the distributive law, we define a new operad, which is called a Lie-Leibniz operad. The Lie-Leibniz operad is Koszul, because the Lie operad and the Leibniz operad are both Koszul. The Koszul dual operad of Lie-Leibniz operad is also studied. In addition, we will prove that the Lie-Leibniz operad naturally becomes a resolution over the Lie operad.

1 Introduction

Leibniz algebras of Loday [8, 9] and their homotopy versions are the key concepts in higher dimensional topological field theory. Hence it is interesting work to study the operad of Leibniz algebras (See Ginzburg-Kapranov [3], Loday [9] and so on, for the details of operads).

It is known that the operad of Leibniz algebras $\mathcal{L}eib$ is isomorphic to $\mathcal{L}ie\otimes\mathcal{P}erm$ (cf. Vallette [15]), where $\mathcal{L}ie$ is the operad of Lie algebras and $\mathcal{P}erm$ is the one of permutation algebras of Chapoton [2]. This implies that between Lie operad and Leibniz operad there are two natural binary-quadratic (bq) relations,

$$[1,(2,3)] = ([1,2],3) + (2,[1,3]), \tag{1}$$

$$[(1,2),3] = ([1,2],3) - ([2,1],3), \tag{2}$$

where $(1,2)$ is the generator of Lie operad, $[1,2]$ and $[2,1]$ are the generators of Leibniz operad. We call a Leibniz algebra a **Lie-Leibniz algebra** (or Lie-Loday algebra), if it has a Lie bracket satisfying (1) and (2). These relations naturally arise in **derived bracket construction** (Kosmann-Schwarzbach [6, 7], see also

[14]). Hence we call (1) and (2) the natural relations[1].

We will prove that the natural relations (1) and (2) are **distributive laws** of $\mathcal{L}ie$ over $\mathcal{L}eib$ in the sense of Markl [10] and Beck [1]. In [10] it was proved that if two Koszul operads are unified by distributive laws, then the deduced operad is again Koszul. Because $\mathcal{L}ie$ and $\mathcal{L}eib$ are both Koszul, the operad of Lie-Leibniz algebras, which is denoted by $\mathcal{LL}$, is Koszul. We also prove that by the derived bracket construction the operad $\mathcal{LL}$ can be decomposed into the form of tensor product

$$\mathcal{LL} = \mathcal{L}ie \otimes \mathcal{D}$$

where $\mathcal{D}$ is a new operad defined in Section 2.2.

We shortly describe a connection between Lie-Leibniz algebras and Courant algebroids, or 3-dimensional topological field theory of AKSZ type (cf. Ikeda [5], see also Roytenberg [12]). The Courant algebroids are defined as vector bundles over smooth manifolds, $B \to M$, equipped with Leibniz brackets $[\cdot,\cdot]$ of degree -1, symplectic structures $(\cdot,\cdot)$ of degree -2, and derivation representations $\rho : B \to TM$ of degree -1, satisfying,

$$\begin{aligned}
[x_1,[x_2,x_3]] &= [[x_1,x_2],x_3] + [x_2,[x_1,x_3]], &(3)\\
\rho(x_1)(x_2,x_3) &= ([x_1,x_2],x_3) + (x_2,[x_1,x_3]), &(4)\\
\rho(x_3)(x_1,x_2) &= ([x_1,x_2],x_3) + ([x_2,x_1],x_3), &(5)
\end{aligned}$$

where x_1, x_2, x_3 are smooth sections of the bundle and where the degrees of variables are all $+1$. By (3), the space of sections becomes a (graded-)Leibniz algebra. Because the product $(\cdot,\cdot)$ is a symplectic structure, i.e., Poisson bracket, it is a Lie bracket. In (5), because the degrees of variables are $+1$,

$$([1,2] - [2,1])(x_1,x_2) = [x_1,x_2] + [x_2,x_1].$$

Because $\rho(x)$ is a derivation, the left-hand sides of (4) and (5) are regarded as Leibniz brackets, $\rho(x_1)(x_2,x_3) = [x_1,(x_2,x_3)]$ and $\rho(x_3)(x_1,x_2) = [(x_1,x_2),x_3]$. In fact, (4) and (5) are geometric examples of the distributive laws (1) and (2). Thus we get a slogan:

Courant brackets are distributive laws

[1] The relation (1) arises in other context, i.e., the study of tri-algebras. See Gubarev-Kolesnikov's work [4] for the detailed study of this direction.

This provides a new picture for the study of Courant algebroids.

In [11] (see also [13]), Roytenberg and Weinstein studied the skewsymmetrized bracket of Courant bracket,

$$\{x_1, x_2\} := \frac{1}{2}\Big([x_1, x_2] - [x_2, x_1]\Big).$$

This bracket satisfies a Jacobi identity *up to homotopy*,

$$\{\{x_1, x_2\}, x_3\} + \{\{x_3, x_1\}, x_2\} + \{\{x_2, x_3\}, x_1\} = \mathbf{d}T(x_1, x_2, x_3), \tag{6}$$

where $\mathbf{d}$ is a differential, which is defined as the dual of ρ above, and the Jacobi anomaly T has the following form,

$$T(x_1, x_2, x_3) := \frac{1}{3}\Big(([x_1, x_2], x_3) + ([x_3, x_1], x_2) + ([x_2, x_3], x_1)\Big).$$

It has been proved that Courant algebroids have special strong-homotopy (sh) Lie algebra structures. We will prove that the operad of Lie-Leibniz algebras is a resolution over the Lie-operad. This proposition provides an operad theoretical description for Roytenberg-Weinstein's sh-Lie algebra theory.

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2 Main results

ASSUMPTIONS.

The ground field is $\mathbb{K}$ of $char(\mathbb{K}) := 0$ and $\mathbb{Q} \subset \mathbb{K}$.

The Leibniz algebras are left.

The operads $\mathcal{P}$ are algebraic and $\mathcal{P}(1) := \mathbb{K}$.

2.1 Distributive laws

In this subsection we recall distributive laws of Markl [10].

Let $\mathcal{P} = (E_{\mathcal{P}}, R_{\mathcal{P}})$ and $\mathcal{Q} = (E_{\mathcal{Q}}, R_{\mathcal{Q}})$ be the binary-quadratic (bq) operads, where $E.$ and $R.$ are respectively the space of generators and the one of bq-relations. We consider a linear mapping,

$$\delta : \mathcal{Q}(2) \odot \mathcal{P}(2) \Rightarrow \mathcal{P}(2) \odot \mathcal{Q}(2), \tag{7}$$

where $\odot$ is the grafting of trees[2]. Let $\Delta :=< x - \delta(x) >$ be the graph of the map. One defines a new bq-operad by

$$\mathcal{P}\mathcal{Q} := (E_{\mathcal{P}} \oplus E_{\mathcal{Q}}, R_{\mathcal{P}} \oplus \Delta \oplus R_{\mathcal{Q}}).$$

[2]The arity of $\mathcal{Q}(2) \odot \mathcal{P}(2)$ is 3.

We assume that the degree of $E_{\mathcal{Q}}$ is $+1$. Then $\mathcal{PQ}$ becomes a graded operad, in particular,

$$\mathcal{PQ}(4) = \mathcal{PQ}^0(4) \oplus \mathcal{PQ}^1(4) \oplus \mathcal{PQ}^2(4) \oplus \mathcal{PQ}^3(4).$$

It is easy to see that $\mathcal{PQ}^0(4) \cong \mathcal{P}(4)$ and $\mathcal{PQ}^3(4) \cong \mathcal{Q}(4)$. The map δ or the relation Δ is called a **distributive law**, if naturally

$$(\overline{\mathcal{P}} \odot \mathcal{Q})(4) \cong \mathcal{PQ}^1(4) \oplus \mathcal{PQ}^2(4), \tag{8}$$

where $\overline{\mathcal{P}} := (\mathcal{P}(2), \mathcal{P}(3))$. There exists a natural homogeneous epimorphism,

$$epi : (\overline{\mathcal{P}} \odot \mathcal{Q})(4) \to \mathcal{PQ}^1(4) \oplus \mathcal{PQ}^2(4),$$

which is defined by the universality of the grafting product. Hence the above definition says that this map is mono.

Example 2.1. *Let $\mathcal{P} = \mathcal{C}om$ be the commutative associative operad and let $\mathcal{Q} = \mathcal{L}ie$ the Lie operad. Then the derivation condition*

$$\begin{aligned} \mathcal{L}ie(2) \odot \mathcal{C}om(2) &\Rightarrow \mathcal{C}om(2) \odot \mathcal{L}ie(2), \\ [1, 23] &\Rightarrow [1, 2]3 + 2[1, 3] \end{aligned}$$

defines a distributive law. The induced operad $\mathcal{C}om\mathcal{L}ie (=: \mathcal{P}oiss)$ is the Poisson operad.

2.2 Lie-Leibniz algebras

We consider a 3-dimensional vector space generated by the binary trees,

$$< [1, 2] \ , \ [1, 2]_t \ , \ (1, 2) > .$$

An S_2-module structure is defined on the 3-space by

$$\begin{aligned} [1, 2]_t &= [2, 1], \\ (1, 2) &= -(2, 1). \end{aligned}$$

The operad of Lie-Leibniz algebras, which is denoted by $\mathcal{LL}$, is generated over the 3-space, whose bq-relations are

$$[1, [2, 3]] - [[1, 2], 3] - [2, [1, 3]], \tag{9}$$

$$[1, (2, 3)] - ([1, 2], 3) - (2, [1, 3]), \tag{10}$$

$$[(1, 2), 3] - ([1, 2], 3) + ([2, 1], 3), \tag{11}$$

$$(1, (2, 3)) - ((1, 2), 3) - (2, (1, 3)), \tag{12}$$

or equivalently,

$$[[2,3],1]_t - [[1,2],3] - [[3,1]_t,2]_t, \tag{13}$$
$$[(2,3),1]_t - ([1,2],3) + ([3,1]_t,2), \tag{14}$$
$$[(1,2),3] - ([1,2],3) + ([1,2]_t,3), \tag{15}$$
$$((1,2),3) + ((3,1),2) + ((2,3),1), \tag{16}$$

The formulas (13)-(16) will be used in the next section.

Definition 2.2. *The algebras over the operad $\mathcal{LL}$ are called the Lie-Leibniz algebras or Lie-Loday algebras. The bidegree of Lie-Leibniz algebra is (i,j), if the degree of Lie bracket is i and the one of Leibniz bracket is j.*

Example 2.3 ([6, 7])**.** *Let $(\mathfrak{g},(,),d)$ be a dg-Lie algebra with a Lie bracket $(,)$ of degree 0 and a differential of degree $+1$. Define a Leibniz bracket by*

$$[x,y] := -(-1)^{|x|}(dx,y).$$

Then $(\mathfrak{g},(,),[,])$ becomes a graded Lie-Leibniz algebra of bidegree $(0,1)$.

The Leibniz bracket in above example is called a **derived bracket**. In [14] it has been proved that on the level of operad the universal Leibniz bracket is given as the derived bracket. That is, the generators of Leibniz operad $[1,2]$ and $[1,2]_t(=[2,1])$ are the derived brackets of the one of Lie operad,

$$\begin{aligned} [1,2] &= (d1,2), \\ [1,2]_t &= -(1,d2). \end{aligned}$$

Here d is a formal differential. The derived bracket construction derives the natural relations (10) and (11), for example (11) is

$$[(1,2),3] = (d(1,2),3) = ((d1,2),3) + ((1,d2),3) = ([1,2],3) - ([2,1],3).$$

In a similar way, (10) is followed from the Jacobi identity of the Lie bracket. From this we notice that the monomials in $\mathcal{LL}$ are represented by the Lie operad and the formal differentials, for example,

$$\begin{aligned} [[1,2],3] &\cong (\pm)((1,2),3) \otimes (d \otimes d \otimes \mathbf{1}), \\ [(1,2),3] &\cong ((1,2),3) \otimes (d \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes d \otimes \mathbf{1}), \\ [1,(2,3)] &\cong (1,(2,3)) \otimes (d \otimes \mathbf{1} \otimes \mathbf{1}), \end{aligned}$$

where $\mathbf{1}$ is the identity and $(\pm)$ is an appropriate sign.

To study the operad $\mathcal{LL}$ we introduce a new operad $\mathcal{D}$ which is called a **deriving operad**. The deriving operad is defined as follows. We consider an S-module $(d, \mathbf{1} \otimes \mathbf{1}, 0, ...)$ and the free operad over the module, $\mathcal{T}(d, \mathbf{1} \otimes \mathbf{1})$, where d is a 1-ary operator and $\mathbf{1} \otimes \mathbf{1}$ is a binary commutative product. Define an operad $\mathcal{O}$ as a quotient operad of $\mathcal{T}(d, \mathbf{1} \otimes \mathbf{1})$,

$$\mathcal{O} := \mathcal{F}(d, \mathcal{C}om(2))/(R_1, R_2)$$

where (R_1, R_2) is the space of two quadratic relations such that R_1 is the differential properties

$$\begin{aligned} dd &= 0, \\ d(\mathbf{1} \otimes \mathbf{1}) - d \otimes \mathbf{1} - \mathbf{1} \otimes d &= 0 \end{aligned}$$

and R_2 is the associative law,

$$(\mathbf{1} \otimes \mathbf{1}) \otimes \mathbf{1} = \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} = \mathbf{1} \otimes (\mathbf{1} \otimes \mathbf{1}).$$

The operad $\mathcal{O}$ becomes a graded operad, $\mathcal{O} = (\mathcal{O}^i)$, whose degree is defined as the number of d.

Definition 2.4 (deriving operad)**.** *The nth component of $\mathcal{D}$ is by definition*

$$\mathcal{D}(n) := \mathcal{O}^0(n) \oplus \cdots \oplus \mathcal{O}^{n-1}(n).$$

It is obvious that $\mathcal{D}^0 \cong \mathcal{C}om$ (the commutative associative operad) and it is easy to prove that $\mathcal{D}^{top} \cong \mathcal{P}erm$, the operad of permutation algebras of Chapton ([2]), for the isomorphism see [14]. In general $\mathcal{D}$ has the following form.

$$\begin{aligned} \mathcal{D}^0 &= \mathcal{C}om, \\ \mathcal{D}^1(2) &= < d \otimes \mathbf{1} \ , \ \mathbf{1} \otimes d >, \\ \mathcal{D}^1(3) &= < d \otimes \mathbf{1} \otimes \mathbf{1} \ , \ \mathbf{1} \otimes d \otimes \mathbf{1} \ , \ \mathbf{1} \otimes \mathbf{1} \otimes d >, \\ \mathcal{D}^2(3) &= < d \otimes d \otimes \mathbf{1} \ , \ d \otimes \mathbf{1} \otimes d \ , \ \mathbf{1} \otimes d \otimes d >, \\ \cdots &= \cdots . \end{aligned}$$

From this observation, we obtain

Proposition 2.5. *$\mathcal{LL} \cong \mathcal{L}ie \otimes \mathcal{D}$.*

From $\dim \mathcal{L}ie(n) = (n-1)!$ and $\dim \mathcal{C}om(n) = 1$, we obtain

$$\dim \mathcal{LL}(n) = \sum_{m=1}^{n} (n-1)! \binom{n}{m}.$$

Corollary 2.6. *Let $R_{\mathcal{LL}}$ be the space of quadratic relations of $\mathcal{LL}$. Then* $\dim R_{\mathcal{LL}} =$ 13.

Proof. It is known that for any bq-operad $\mathcal{P} = (E, R)$,

$$3(\dim E)^2 = \dim \mathcal{P}(3) + \dim R.$$

In the case of $\mathcal{LL}$, $\dim E = 3$ and $\dim \mathcal{LL}(3) = 14$, which gives the identity of the lemma. □

The natural relations (10) and (11) define a mapping

$$\mathcal{L}eib(2) \odot \mathcal{L}ie(2) \Rightarrow \mathcal{L}ie(2) \odot \mathcal{L}eib(2). \tag{17}$$

We should prove that this is a distributive law.

Theorem 2.7. *The natural relations (10) and (11) define distributive laws.*

Proof. The degree of $\mathcal{L}eib(2)$ is by assumption $+1$. Then $\mathcal{LL}(4)$ becomes a graded space which is decomposed into the 4 subspaces

$$\mathcal{LL}(4) = \mathcal{L}ie(4) \oplus \mathcal{LL}^1(4) \oplus \mathcal{LL}^2(4) \oplus \mathcal{L}eib(4).$$

From Proposition 2.5 we obtain

$$\dim \mathcal{LL}^1(4) = 24, \tag{18}$$
$$\dim \mathcal{LL}^2(4) = 36. \tag{19}$$

To prove the theorem it suffices to show that the dimension of $(\overline{\mathcal{L}ie} \odot \mathcal{L}eib)(4)$ is equal to $60 (= 24 + 36)$, where $\overline{\mathcal{L}ie} = (\mathcal{L}ie(2), \mathcal{L}ie(3))$.

We compute the dimension of $(\overline{\mathcal{L}ie} \odot \mathcal{L}eib)(4)$. The subspace of $(\overline{\mathcal{L}ie} \odot \mathcal{L}eib)(4)$ of degree $+2$ is

$$\mathcal{L}ie(2) \odot \Big(\mathcal{L}eib(2) \otimes \mathcal{L}eib(2)\Big) \oplus \mathcal{L}ie(2) \odot \mathcal{L}eib(3). \tag{20}$$

The first term in (20) is generated by the 12-monomials,

$$\begin{array}{ccc}
([1,2],[3,4]) & ([1,3],[2,4]) & ([1,4],[2,3]) \\
([2,1],[3,4]) & ([3,1],[2,4]) & ([4,1],[2,3]) \\
([1,2],[4,3]) & ([1,3],[4,2]) & ([1,4],[3,2]) \\
([2,1],[4,3]) & ([3,1],[4,2]) & ([4,1],[3,2]).
\end{array}$$

Hence we obtain

$$\dim \mathcal{L}ie(2) \odot \Big(\mathcal{L}eib(2) \otimes \mathcal{L}eib(2)\Big) = 12.$$

The second term of (20) is generated by the generators of 4-types,

$$(1, \mathcal{L}eib(3)) \quad (2, \mathcal{L}eib(3)) \quad (3, \mathcal{L}eib(3)) \quad (4, \mathcal{L}eib(3)).$$

It is known that $\mathcal{L}eib(n) \cong S_n$. Therefore $\dim \mathcal{L}eib(3) = 6$, which gives

$$\dim \mathcal{L}ie(2) \odot \mathcal{L}eib(3) = 4 \times 6 = 24.$$

Thus we obtain

$$\dim \mathcal{L}ie(2) \odot \Big(\mathcal{L}eib(2) \otimes \mathcal{L}eib(2)\Big) \oplus \mathcal{L}ie(2) \odot \mathcal{L}eib(3) = 12 + 24 = 36.$$

This number coincides with the dimension of $\mathcal{LL}_2(4)$.

We consider the subspace of $(\overline{\mathcal{L}ie} \odot \mathcal{L}eib)(4)$ of degree $+1$, which has the form

$$\mathcal{L}ie(3) \odot \mathcal{L}eib(2).$$

Up to the Jacobi identity on $\mathcal{L}ie(3)$, this subspace is generated by

$$\begin{array}{cc}
(([1,2],3),4) & ((4,[1,2]),3) \\
(([2,1],3),4) & ((4,[2,1]),3) \\
(([1,3],2),4) & ((4,[1,3]),2) \\
(([3,1],2),4) & ((4,[3,1]),2) \\
(([1,4],2),3) & ((3,[1,4]),2) \\
(([4,1],2),3) & ((3,[4,1]),2) \\
\cdots & \cdots,
\end{array}$$

totally 24 terms, this number is the same as the dimension of $\mathcal{LL}_1(4)$. $\square$

By the theorems in [10], we obtain the two corollaries below.

Corollary 2.8. *The operad $\mathcal{LL}$ is Koszul for any bidegree (i, j).*

Corollary 2.9. *Given a Leibniz algebra $\mathfrak{g}$, the free Lie algebra over $\mathfrak{g}$, $\mathcal{F}_{\mathcal{L}ie}(\mathfrak{g})$, becomes the free Lie-Leibniz algebra in the category of Leibniz algebras. In particular, when $\mathfrak{g}$ is free, it is the free Lie-Leibniz algebra.*

Remark 2.10 (Poisson embedding)**.** *Since the free Poisson algebra is defined by $\mathcal{F}_{\mathcal{P}oiss} := \mathcal{F}_{\mathcal{C}om}\mathcal{F}_{\mathcal{L}ie}$, we obtain $\mathfrak{g} \subset \mathcal{F}_{\mathcal{P}oiss}(\mathfrak{g})$. An arbitrary Leibniz algebra can be embedded in the free Poisson algebra and the Leibniz bracket is compatible with the Poisson bracket.*

3 Koszul dual algebras

We have proved $\mathcal{LL} = \mathcal{Lie}\mathcal{Leib}$. It is known that the dual of distributive law (i.e. the dual map δ^*) is again a distributive law. Hence the Koszul dual of $\mathcal{Lie}\mathcal{Leib}$ is $\mathcal{Zinb}\mathcal{Com}$, where $\mathcal{Com} = \mathcal{Lie}^!$ and $\mathcal{Zinb} = \mathcal{Leib}^!$ (the Zinbiel operad [16]). The space of generators of $\mathcal{Zinb}\mathcal{Com}$ is a 3-space,

$$< 1 * 2\ ,\ 1 *_t 2\ ,\ 1 \cdot 2 >,$$

where $1 * 2$ and $1 \cdot 2$ are respectively the duals of $[1, 2]$ and $(1, 2)$. The S_2-module structure is as follows,

$$\begin{aligned} 1 *_t 2 &= 2 * 1, \\ 1 \cdot 2 &= 2 \cdot 1. \end{aligned}$$

Proposition 3.1. *The quadratic relations of $\mathcal{Zinb}\mathcal{Com}$ are*

$$1 * (2 * 3) = (1 * 2) * 3 + (2 * 1) * 3, \tag{21}$$

$$(1 * 2) \cdot 3 = 1 * (2 \cdot 3) - (1 \cdot 2) * 3, \tag{22}$$

$$1 \cdot (2 \cdot 3) = (1 \cdot 2) \cdot 3. \tag{23}$$

The relation (22) defines a distributive law of $\mathcal{Zinb}$ over $\mathcal{Com}$.

In [9] it has been proved that the Koszul dual of Leibniz identity is (21).

Proof. Let R' be the space of relations generated by (21), (22) and (23). The relation (21) generates 6-basis, (23) generates 2-basis and (22) generates the following 6-basis,

$$\begin{aligned} (1 * 2) \cdot 3 &= 1 * (2 \cdot 3) - (1 \cdot 2) * 3, \\ (2 * 1) \cdot 3 &= 2 * (1 \cdot 3) - (2 \cdot 1) * 3, \\ (2 * 3) \cdot 1 &= 2 * (3 \cdot 1) - (2 \cdot 3) * 1, \\ (3 * 2) \cdot 1 &= 3 * (2 \cdot 1) - (3 \cdot 2) * 1, \\ (3 * 1) \cdot 2 &= 3 * (1 \cdot 2) - (3 \cdot 1) * 2, \\ (1 * 3) \cdot 2 &= 1 * (3 \cdot 2) - (1 \cdot 3) * 2. \end{aligned}$$

Thus we obtain

$$\dim R' = 14,$$

which satisfies the consistency condition,

$$\dim \mathcal{LL}(3) = \dim R'.$$

The equation (22) is equivalent to

$$(1*2)\cdot 3-(2\cdot 3)*_t 1+(1\cdot 2)*3. \tag{24}$$

The pairing $<,>$ which defines the Koszul duality is defined as follows,

$$\begin{aligned}
<[[i,j],k],(a*b)*c> &= \delta_{ia}\delta_{jb}\delta_{kc},\\
<[[i,j]_t,k],(a*_t b)*c> &= -\delta_{ia}\delta_{jb}\delta_{kc},\\
<[[i,j],k]_t,(a*b)*_t c> &= -\delta_{ia}\delta_{jb}\delta_{kc},\\
<[[i,j]_t,k]_t,(a*_t b)*_t c> &= \delta_{ia}\delta_{jb}\delta_{kc},\\
<([i,j]_t,k),(a*_t b)\cdot c> &= -\delta_{ia}\delta_{jb}\delta_{kc},\\
<([i,j],k),(a*b)\cdot c> &= \delta_{ia}\delta_{jb}\delta_{kc},\\
<[(i,j),k],(a\cdot b)*c> &= \delta_{ia}\delta_{jb}\delta_{kc},\\
<[(i,j),k]_t,(a\cdot b)*_t c> &= -\delta_{ia}\delta_{jb}\delta_{kc},\\
<((i,j),k),(a\cdot b)\cdot c> &= \delta_{ia}\delta_{jb}\delta_{kc},
\end{aligned}$$

and all others zero, where δ is Kronecker's delta and $(ijk),(abc)\in\{(123),(312),(231)\}$. By a direct computation, for example

$$<(13),(24)>=<(14),(24)>=<(15),(24)>=<(16),(24)>=0,$$

one can show that R' is the orthogonal space of $R_{\mathcal{LL}}$, i.e., $R'=R^{\perp}_{\mathcal{LL}}$, with respect to the pairing. $\square$

4 Jacobi anomaly in $\mathcal{LL}$

In this section we study a homotopy theory associated with $\mathcal{LL}$. Suppose that the degree of $(1,2)$ is 0 and the one of $[1,2]$ is $+1$. The second assumption equivalently means that the formal differential in the derived bracket $[1,2]=(d1,2)$ has the degree $+1$. We denote by $\mathbf{s}\mathcal{P}$ the shifted operad of $\mathcal{P}$. The Leibniz operad of this section is $\mathbf{s}\mathcal{L}eib$.

We notice that there exists a natural differential operator, which is denoted by $\mathbf{d}$, on the deriving operad $\mathcal{D}$.

Definition 4.1. *For any $l_1\otimes\cdots\otimes l_n\in\mathcal{D}^j(n)$, $j\le n-2$,*

$$\mathbf{d}(l_1\otimes\cdots\otimes l_n):=\sum_i(\pm)l_1\otimes\cdots\otimes d(l_i)\otimes\cdots\otimes l_n$$

where $d(\mathbf{1})=d$ and $(\pm)$ is an appropriate sign.

For example, $\mathbf{d}(\mathbf{1} \otimes d \otimes \mathbf{1}) = d \otimes d \otimes \mathbf{1} - \mathbf{1} \otimes d \otimes d$.

Remark 4.2. *We should remark that* $(\mathcal{D}, \mathbf{d})$ *is not a dg-operad.*

It is an easy exercise to prove that $(\mathcal{D}, \mathbf{d})$ is a resolution of $\mathbf{s}\mathcal{C}om$, i.e.,

$$\mathcal{D}^{top}/\mathrm{Im}\mathbf{d} \cong \mathbf{s}\mathcal{C}om$$

and $\mathrm{H}^{\cdot}(\mathcal{D}, \mathbf{d}) = 0$. From the identities $\mathcal{LL} \cong \mathcal{L}ie \otimes \mathcal{D}$ and $\mathcal{L}ie \otimes \mathbf{s}\mathcal{C}om \cong \mathbf{s}\mathcal{L}ie$, we obtain

Proposition 4.3. *The complex* $(\mathcal{LL}, 1 \otimes \mathbf{d})$ *is a resolution of* $\mathbf{s}\mathcal{L}ie$.

Eq. (6) in Introduction is considered to be a representation of this proposition. The tensor T lives in $\mathcal{LL}^1(3)$.

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